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Description of the Sigma-coordinate transform in the EPISODE model

Old and new version

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The sigma-coordinate transform presently applied in EPISODE

The vertical extent of the model domain of the urban dispersion model EPISODE is defined from the ground and up to a constant height, H, above ground. This means that the model applies a stretched vertical coordinate, or a sigma-coordinate system, given by the following transformation:

$$x^* = x,\quad y^* = y,\quad \sigma = \sigma(x,y,z) = z - h(x,y) \tag{A}$$

In this transformed coordinate system, and with the additional assumptions of an incompressible windfield and of a simplified parameterization of the terms describing the horizontal turbulent diffusion, the advection/diffusion equation in EPISODE is given by

$$\begin{aligned}\frac{\partial c_i}{\partial t} + \frac{\partial}{\partial x^*}(uc_i) + \frac{\partial}{\partial y^*}(vc_i) + \frac{\partial}{\partial \sigma}(\omega c_i) &= \frac{\partial}{\partial x^*}\left(K^{(H)}\frac{\partial c_i}{\partial x^*}\right) + \frac{\partial}{\partial y^*}\left(K^{(H)}\frac{\partial c_i}{\partial y^*}\right)\\ &+ \frac{\partial}{\partial \sigma}\left(K^{(Z)}\frac{\partial c_i}{\partial \sigma}\right) + R_i - S_i\end{aligned} \tag{B}$$

where c_i is the species concentration, u and v are the horizontal components of the wind velocity, and $K^{(H)}$ and $K^{(Z)}$ are the horizontal and vertical eddy diffusivities, respectively. R_i is the source terms and S_i represents the sinks.

The vertical velocity in the transformed system, ω, is defined by

$$\omega \equiv w - u\frac{\partial h}{\partial x} - v\frac{\partial h}{\partial y}\ , \tag{C}$$

where w is the wind component in the (Cartesian) vertical direction.

The wind field is assumed incompressible and satisfy the continuity equation

$$\frac{\partial u}{\partial x^*} + \frac{\partial v}{\partial y^*} + \frac{\partial \omega}{\partial \sigma} = 0 \tag{D}$$

Alternative sigma-coordinate transform to be applied in EPISODE

The vertical extent of the model domain is changed so that the model is defined from the ground and up to a constant height, H_0, above sea level. This means that the model applies a stretched vertical coordinate, or a sigma-coordinate system, given by the following transformation:

$$x = x^*,\quad y = y^*,\quad \sigma = \sigma(x,y,z) = H_0\frac{z - h(x,y)}{H_0 - h(x,y)} \tag{E}$$

Note that denominator of eq. (E) is identical to the total vertical depth of the model, i.e. D(x,y) defined as:

$$D(x,y) \equiv H_0 - h(x,y) \quad \Leftrightarrow \quad h(x,y) = H_0 - D(x,y)$$

From (E) it is seen that a grid volume element: $\delta V = \delta x \cdot \delta y \cdot \delta z = \frac{D(x,y)}{H_0} \delta x^* \cdot \delta y^* \cdot \delta\sigma$.

With the definition (E): $\sigma = 0$ for $z = h(x,y)$ and $\sigma = H_0$ for $z = H_0$, i.e. $\sigma \in [0, H_0]$.

Note also that:

$$\frac{\partial h}{\partial \xi} = -\frac{\partial D}{\partial \xi}$$

where ξ is either x or y.

In this transformed coordinate system, and with the same additional assumptions as above (i.e. incompressible windfield and simplified parameterization of the terms describing the horizontal turbulent diffusion), the advection/diffusion equation in EPISODE becomes

$$\frac{\partial c_i}{\partial t} + \frac{1}{D}\left(\frac{\partial(uc_iD)}{\partial x^*} + \frac{\partial(vc_iD)}{\partial y^*} + \frac{\partial(\omega c_iD)}{\partial \sigma}\right) = \frac{\partial}{\partial x^*}\left(K^{(H)}\frac{\partial c_i}{\partial x^*}\right) + \frac{\partial}{\partial y^*}\left(K^{(H)}\frac{\partial c_i}{\partial y^*}\right) + \left(\frac{H_0}{D}\right)^2 \frac{\partial}{\partial \sigma}\left(K^{(Z)}\frac{\partial c_i}{\partial \sigma}\right) + R_i - S_i \qquad \text{(F)}$$

The new vertical velocity, ω, is defined by

$$\omega \equiv \frac{H_0}{D} w - (H_0 - \sigma)\frac{u}{D}\frac{\partial h}{\partial x} - (H_0 - \sigma)\frac{v}{D}\frac{\partial h}{\partial y}, \qquad \text{(G)}$$

and the incompressible wind field satisfy the continuity equation

$$\frac{\partial(uD)}{\partial x^*} + \frac{\partial(vD)}{\partial y^*} + \frac{\partial(\omega D)}{\partial \sigma} = 0. \qquad \text{(H)}$$

Description of the Sigma-coordinate transform in the EPISODE model

Old and new version

1 Introduction

This report describes both the old sigma-coordinate transform that has been applied in earlier versions of the Eulerian dispersion model EPISODE, and a new transform that recently has been implemented. In this report the two transforms are described in detail. Transforms of this type is introduced in order to simplify the boundary conditions. Instead of specifying boundary conditions for a complex surface given by z = h(x,y), the tranformed equations just need the conditions on a simple coordinate surface, σ = constant. Transforms of these types are applied in a variety of analytical and numerical models. A general description of coordinate transforms can be found in chapter 6; pp 102-127, of Pielke (1984).

2 The old σ-coordinate transform applied in EPISODE

The vertical extent of the model domain of the urban dispersion model EPISODE is defined from the ground and up to a constant height, H, above ground. This means that the model applies a stretched vertical coordinate, or a sigma-coordinate system, given by the following transformation:

$$x^* = x^*(x,y,z) = x \tag{1a}$$

$$y^* = y^*(x,y,z) = y \tag{1b}$$

$$\sigma = \sigma(x,y,z) = z - h(x,y) \tag{1c}$$

where h(x,y) is the height of the ground above mean sea level. The physical extent of the model domain when applying the transform (1 a-c) is depicted in Figure 1.

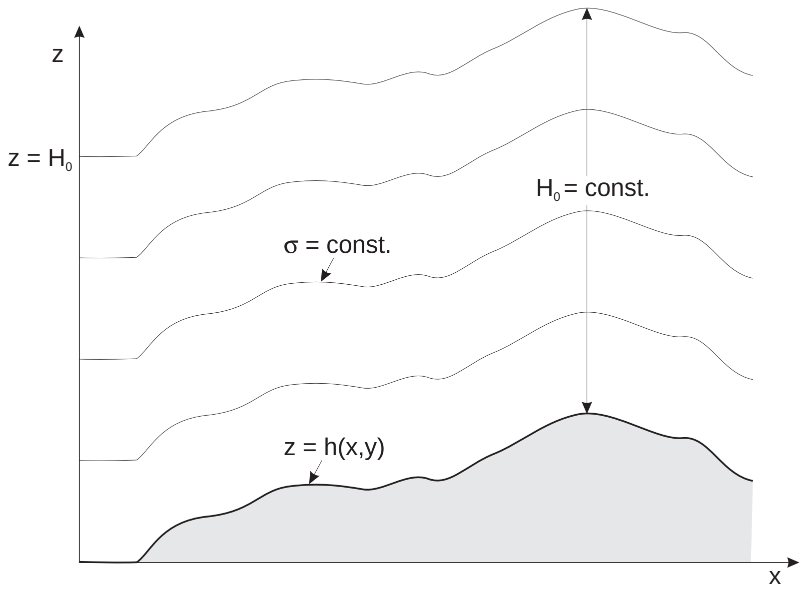

Figure 1: A schematic representation of the vertical extent of the model domain and the position of the model layers, when the transform of (1 a-c) is applied.

As seen in Figure 1 the model levels maintain the shape of the topography at all heights within the model domain. In reality the influence of the topography on the wind field will decrease with height and the wind will become more and more horizontal with height. In the present σ-transform such horizontal flow might induce considerable vertical transport between the σ-layers. Besides, if the dispersion model is supplied with wind data from an off-line numerical weather prediction model (MM5 or others), severe interpolation will be necessary at the open boundaries in order to match the coordinate systems of the two models. This is so because the vast majority (if not all) of the NWP-models apply a vertical coordinate that levels out with height.

Coordinate transforms of this type leads to changes in the expressions for the derivatives. Since the transform is independent of time, only spatial derivatives are influenced. Applying the chain-rule the following relations between the derivatives in the ordinary Cartesian system (x, y, z) and the transformed coordinate system (x^*, y^*, σ) is found:

$$\frac{\partial}{\partial x} = \frac{\partial x^*}{\partial x}\frac{\partial}{\partial x^*} + \frac{\partial y^*}{\partial x}\frac{\partial}{\partial y^*} + \frac{\partial \sigma}{\partial x}\frac{\partial}{\partial \sigma} \tag{2a}$$

$$\frac{\partial}{\partial y} = \frac{\partial x^*}{\partial y}\frac{\partial}{\partial x^*} + \frac{\partial y^*}{\partial y}\frac{\partial}{\partial y^*} + \frac{\partial \sigma}{\partial y}\frac{\partial}{\partial \sigma} \tag{2b}$$

$$\frac{\partial}{\partial z} = \frac{\partial x^*}{\partial z}\frac{\partial}{\partial x^*} + \frac{\partial y^*}{\partial z}\frac{\partial}{\partial y^*} + \frac{\partial \sigma}{\partial z}\frac{\partial}{\partial \sigma} \tag{2c}$$

Applying the definitions in 1a-c we then get:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x^*} - \frac{\partial h}{\partial x}\frac{\partial}{\partial \sigma} \tag{3a}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial y^*} - \frac{\partial h}{\partial y}\frac{\partial}{\partial \sigma} \tag{3b}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial \sigma} \tag{3c}$$

NB: Functions, f(x,y), that are not explicitly dependent on the vertical coordinate are treated identically in the two coordinate systems. This is seen by application of eqs. 3 a-c:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x^*} - \frac{\partial h}{\partial x}\frac{\partial f}{\partial \sigma} = \frac{\partial f}{\partial x^*} \quad ; \quad \frac{\partial f}{\partial y} = \frac{\partial f}{\partial y^*} - \frac{\partial h}{\partial y}\frac{\partial f}{\partial \sigma} = \frac{\partial f}{\partial y^*} \text{ and } \quad \frac{\partial f}{\partial z} = \frac{\partial f}{\partial \sigma} = 0$$

The advection/diffusion equation(s) that are solved in EPISODE are given by

$$\frac{\partial c_i}{\partial t} + \frac{\partial}{\partial x}(uc_i) + \frac{\partial}{\partial y}(vc_i) + \frac{\partial}{\partial z}(wc_i) = \frac{\partial}{\partial x}\left(K^{(H)}\frac{\partial c_i}{\partial x}\right) + \frac{\partial}{\partial y}\left(K^{(H)}\frac{\partial c_i}{\partial y}\right) + \frac{\partial}{\partial z}\left(K^{(Z)}\frac{\partial c_i}{\partial z}\right) + R_i - S_i \tag{4}$$

where c_i is the species concentration, u, v and w are the three components of the wind velocity, and $K^{(H)}$ and $K^{(Z)}$ are the horizontal and vertical eddy diffusivities, respectively. R_i is the source terms and S_i represents the sinks.

In addition the wind velocity field, which is applied in eq. 4, is assumed incompressible, i.e. the continuity equation becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0\ . \qquad \text{(alternatively} \qquad \nabla\cdot\vec{V}=0\text{)} \tag{5}$$

In the transformed coordinate system eq. 5 can be expressed as

$$\frac{\partial u}{\partial x^*}-\frac{\partial h}{\partial x}\frac{\partial u}{\partial \sigma}+\frac{\partial v}{\partial y^*}-\frac{\partial h}{\partial y}\frac{\partial v}{\partial \sigma}+\frac{\partial w}{\partial \sigma}=0$$

$$\Downarrow$$

$$\frac{\partial u}{\partial x^*}+\frac{\partial v}{\partial y^*}+\frac{\partial}{\partial \sigma}\left(w-u\frac{\partial h}{\partial x}-v\frac{\partial h}{\partial y}\right)=0\ . \tag{6}$$

By defining a new vertical velocity ω given by

$$\omega\equiv w-u\frac{\partial h}{\partial x}-v\frac{\partial h}{\partial y}=\vec{V}\cdot\left(\vec{k}-\frac{\partial h}{\partial x}\vec{i}-\frac{\partial h}{\partial y}\vec{j}\right), \tag{7}$$

where $\vec{V}=u\vec{i}+v\vec{j}+w\vec{k}$ is the Cartesian wind velocity vector, eq. (6) can be expressed in the more compact form

$$\frac{\partial u}{\partial x^*}+\frac{\partial v}{\partial y^*}+\frac{\partial \omega}{\partial \sigma}=0\ . \tag{8}$$

By insertion it is easily verified that $\nabla\sigma=\left(\vec{k}-\frac{\partial h}{\partial x}\vec{i}-\frac{\partial h}{\partial y}\vec{j}\right)$, where ∇ is the Cartesian gradient operator. This vector is directed normal to any of the model layer surfaces, pointing upwards into the air. The direction into the air is evident since the vertical direction of this vector is always positive.

By dividing eq. (7) by the length of this normal vector we define a new quantity, $\dot{\sigma}$, given by

$$\dot{\sigma}\equiv\frac{\omega}{\sqrt{1+\left(\frac{\partial h}{\partial x}\right)^2+\left(\frac{\partial h}{\partial y}\right)^2}}=\frac{\vec{V}\cdot\left(\vec{k}-\nabla h\right)}{\sqrt{1+\left(\frac{\partial h}{\partial x}\right)^2+\left(\frac{\partial h}{\partial y}\right)^2}}=\vec{V}\cdot\vec{n}_{z=\sigma+h(x,y)}\ , \tag{9}$$

which define the component of the wind velocity normal to the model surfaces. Transformation of the different terms in eq. (4) gives

$$\frac{\partial c_i}{\partial t}\Rightarrow\frac{\partial c_i}{\partial t}$$

$$\frac{\partial}{\partial x}\left(uc_i\right)\Rightarrow\frac{\partial}{\partial x^*}\left(uc_i\right)-\frac{\partial h}{\partial x}\frac{\partial\left(uc_i\right)}{\partial \sigma}$$

$$\frac{\partial}{\partial y}(vc_i) \Rightarrow \frac{\partial}{\partial y^*}(vc_i) - \frac{\partial h}{\partial y}\frac{\partial (vc_i)}{\partial \sigma}$$

$$\frac{\partial}{\partial z}(wc_i) \Rightarrow \frac{\partial}{\partial \sigma}(wc_i)$$

$$\frac{\partial}{\partial x}\left(K^{(H)}\frac{\partial c_i}{\partial x}\right) \Rightarrow \frac{\partial}{\partial x^*}\left(K^{(H)}\left(\frac{\partial c_i}{\partial x^*} - \frac{\partial h}{\partial x}\frac{\partial c_i}{\partial \sigma}\right)\right) - \frac{\partial h}{\partial x}\frac{\partial}{\partial \sigma}\left(K^{(H)}\left(\frac{\partial c_i}{\partial x^*} - \frac{\partial h}{\partial x}\frac{\partial c_i}{\partial \sigma}\right)\right)$$

$$\frac{\partial}{\partial y}\left(K^{(H)}\frac{\partial c_i}{\partial y}\right) \Rightarrow \frac{\partial}{\partial y^*}\left(K^{(H)}\left(\frac{\partial c_i}{\partial y^*} - \frac{\partial h}{\partial y}\frac{\partial c_i}{\partial \sigma}\right)\right) - \frac{\partial h}{\partial y}\frac{\partial}{\partial \sigma}\left(K^{(H)}\left(\frac{\partial c_i}{\partial y^*} - \frac{\partial h}{\partial y}\frac{\partial c_i}{\partial \sigma}\right)\right)$$

$$\frac{\partial}{\partial z}\left(K^{(Z)}\frac{\partial c_i}{\partial z}\right) \Rightarrow \frac{\partial}{\partial \sigma}\left(K^{(Z)}\frac{\partial c_i}{\partial \sigma}\right)$$

This means that the left hand side of eq. 4 (i.e. the time derivative and the advection terms) can be expressed in the following way

$$\frac{\partial c_i}{\partial t} + \frac{\partial}{\partial x^*}(uc_i) + \frac{\partial}{\partial y^*}(vc_i) + \frac{\partial}{\partial \sigma^*}(wc_i) - \frac{\partial h}{\partial x}\frac{\partial (uc_i)}{\partial \sigma} - \frac{\partial h}{\partial y}\frac{\partial (vc_i)}{\partial \sigma}$$

$$= \frac{\partial c_i}{\partial t} + \frac{\partial}{\partial x^*}(uc_i) + \frac{\partial}{\partial y^*}(vc_i) + \frac{\partial}{\partial \sigma}\left(wc_i - uc_i\frac{\partial h}{\partial x} - vc_i\frac{\partial h}{\partial y}\right)$$

$$= \frac{\partial c_i}{\partial t} + \frac{\partial}{\partial x^*}(uc_i) + \frac{\partial}{\partial y^*}(vc_i) + \frac{\partial}{\partial \sigma}(\omega c_i)$$

The term in eq. 4 describing the diffusion in the x direction becomes:

$$\frac{\partial}{\partial x^*}\left(K^{(H)}\frac{\partial c_i}{\partial x^*}\right) - \frac{\partial}{\partial x^*}\left(K^{(H)}\frac{\partial h}{\partial x}\frac{\partial c_i}{\partial \sigma}\right) - \frac{\partial h}{\partial x}\frac{\partial}{\partial \sigma}\left(K^{(H)}\frac{\partial c_i}{\partial x^*}\right) + \frac{\partial h}{\partial x}\frac{\partial}{\partial \sigma}\left(K^{(H)}\frac{\partial h}{\partial x}\frac{\partial c_i}{\partial \sigma}\right) \quad (11)$$

Similarly the y-directed diffusion term transforms into:

$$\frac{\partial}{\partial y^*}\left(K^{(H)}\frac{\partial c_i}{\partial y^*}\right) - \frac{\partial}{\partial y^*}\left(K^{(H)}\frac{\partial h}{\partial y}\frac{\partial c_i}{\partial \sigma}\right) - \frac{\partial h}{\partial y}\frac{\partial}{\partial \sigma}\left(K^{(H)}\frac{\partial c_i}{\partial y^*}\right) + \frac{\partial h}{\partial y}\frac{\partial}{\partial \sigma}\left(K^{(H)}\frac{\partial h}{\partial y}\frac{\partial c_i}{\partial \sigma}\right) \quad (12)$$

No additional terms arise as a result of the transformation of the term describing the vertical diffusion,

$$\frac{\partial}{\partial \sigma}\left(K^{(Z)}\frac{\partial c_i}{\partial \sigma}\right) \quad (13)$$

Expressed in the transformed coordinates the advection/diffusion equation, eq. (4), therefore becomes:

$$\frac{\partial c_i}{\partial t} + \frac{\partial}{\partial x^*}(uc_i) + \frac{\partial}{\partial y^*}(vc_i) + \frac{\partial}{\partial \sigma}(\omega c_i) = \frac{\partial}{\partial x^*}\left(K^{(H)}\frac{\partial c_i}{\partial x^*}\right) + \frac{\partial}{\partial y^*}\left(K^{(H)}\frac{\partial c_i}{\partial y^*}\right) +$$

$$\frac{\partial}{\partial \sigma}\left(K^{(Z)}\frac{\partial c_i}{\partial \sigma}\right) + (\text{additional} \cdot \text{diffusion} \cdot \text{terms} \cdot \text{from} \cdot (11) \cdot \text{and} \cdot (12)) + R_i - S_i \quad (14)$$

Where the applied wind velocity components obey:

$$\frac{\partial u}{\partial x^*} + \frac{\partial v}{\partial y^*} + \frac{\partial \omega}{\partial \sigma} = 0 \tag{15}$$

In EPISODE, eq. (14) is simplified by neglecting all the additional terms arising from the transformation of the two terms describing the horizontal diffusion. This equation therefore reduces to the familiar form

$$\begin{aligned} &\frac{\partial c_i}{\partial t} + \frac{\partial}{\partial x^*}(uc_i) + \frac{\partial}{\partial y^*}(vc_i) + \frac{\partial}{\partial \sigma}(\omega c_i) \\ &= \frac{\partial}{\partial x^*}\left(K^{(H)}\frac{\partial c_i}{\partial x^*}\right) + \frac{\partial}{\partial y^*}\left(K^{(H)}\frac{\partial c_i}{\partial y^*}\right) + \frac{\partial}{\partial \sigma}\left(K^{(Z)}\frac{\partial c_i}{\partial \sigma}\right) + R_i - S_i \end{aligned} \tag{16}$$

NOTE: The terms that are neglected in eq. 14 all arises from the transformation of the two expressions originally describing the horizontal diffusion in the cartesian system. However, this does not mean that the simplification from eq. 14 to eq. 16 only affect the horizontal diffusion processes. In eq. 16, the two first terms on the r.h.s. now describe the diffusion along the layer surfaces, and the third term expresses the turbulent diffusion normal to these surfaces.

3 The new σ-coordinate transform presently implemented in EPISODE

The sigma-coordinate transformation described in the previous section keeps the vertical thickness of each layer constant everywhere, see Figure 1. This means that the upper boundary of the model has a form which is identical to the topography below. Physically, however, this boundary should be at an altitude where the influence of the topography below should be of minor importance, i.e. the upper boundary should be above the mixing height. At this height the wind velocity is almost horizontal, and in order to minimize the wind-flow across the upper boundary this boundary should be horizontal as well. The transformation of the vertical coordinate should therefore gradually level out as the vertical distance from the ground surface increases. This can be accomplished by introducing the following sigma coordinate transform:

$$x^* = x^*(x, y, z) = x \tag{17a}$$

$$y^* = y^*(x, y, z) = y \tag{17b}$$

$$\sigma = \sigma(x, y, z) = H_0 \frac{z - h(x, y)}{H_0 - h(x, y)} \tag{17c}$$

where h(x,y) is the height above sea level of the topography, and H_0 is the constant value above sea level of the model height. Note that denominator of eq. (17c) is identical to the total vertical depth of the model, i.e. D(x,y) defined as:

$$D(x, y) \equiv H_0 - h(x, y) \quad \Leftrightarrow \quad h(x, y) = H_0 - D(x, y) \tag{18}$$

The physical extent of the model domain when applying the transform (17 a-c) is depicted in Figure 2.

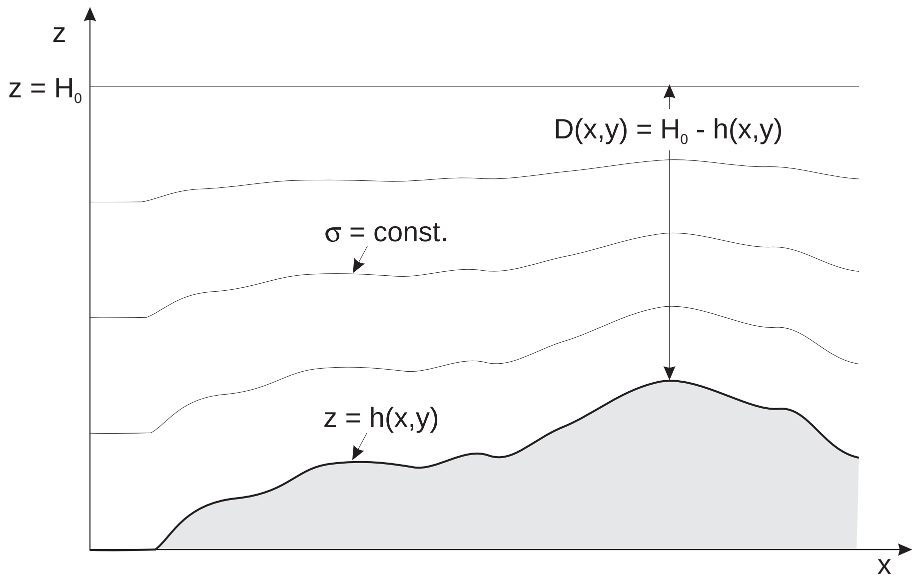

Figure 2: A schematic representation of the vertical extent of the model domain and the position of the model layers, when the transform of (17 a-c) is applied.

From (17 a-c) it is seen that a grid volume element, $\delta V = \delta x \cdot \delta y \cdot \delta z$, in the new co-ordinate system will be given by:

$$\delta V = \frac{D(x,y)}{H_0} \delta x^* \cdot \delta y^* \cdot \delta\sigma .$$

With the definition (17c): $\sigma = 0$ for $z = h(x,y)$ and $\sigma = H_0$ for $z = H_0$, i.e. $\sigma \in [0, H_0]$.

Note also that:

$$\frac{\partial h}{\partial \xi} = -\frac{\partial D}{\partial \xi} \tag{19}$$

where ξ is either x or y. The transformations of the gradients in the Cartesian and the sigma-transformed system are again as given in eqs. (2a), (2b) and (2c). However, since the sigma variable has been changed, the derivatives of σ now becomes:

$$\frac{\partial\sigma}{\partial x} = \frac{\partial}{\partial x}\left(H_0 \frac{z - h(x,y)}{H_0 - h(x,y)} \right) = H_0 \frac{(H_0 - h(x,y)) \frac{\partial}{\partial x}(z - h(x,y)) - (z - h(x,y)) \frac{\partial}{\partial x}(H_0 - h(x,y))}{(H_0 - h(x,y))^2}$$

$$\Downarrow$$

$$\frac{\partial\sigma}{\partial x} = -H_0 \frac{\partial h/\partial x}{H_0 - h(x,y)} + H_0 \frac{z - h(x,y)}{H_0 - h(x,y)} \cdot \frac{\partial h/\partial x}{H_0 - h(x,y)} \tag{20}$$

Utilising the definition of the model depth, D(x,y), and the sigma-coordinate, σ, eq. (20) can be written:

$$\frac{\partial \sigma}{\partial x} = \frac{H_0}{D}\frac{\partial D}{\partial x} - \frac{\sigma}{D}\frac{\partial D}{\partial x} = \frac{(H_0 - \sigma)}{D(x,y)}\frac{\partial D}{\partial x} \tag{21}$$

Similarly we get:

$$\frac{\partial \sigma}{\partial y} = \frac{(H_0 - \sigma)}{D(x,y)}\frac{\partial D}{\partial y} \tag{22}$$

The vertical gradient of the sigma-coordinate becomes simply:

$$\frac{\partial \sigma}{\partial z} = \frac{\partial}{\partial z}\left(H_0 \frac{z - h(x,y)}{H_0 - h(x,y)} \right) = \frac{H_0}{H_0 - h(x,y)} = \frac{H_0}{D(x,y)} \tag{23}$$

Therefore, instead of the transformation equations (3a – 3c), these equations now become:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x^*} + \frac{H_0 - \sigma}{D(x,y)}\frac{\partial D}{\partial x}\frac{\partial}{\partial \sigma} \tag{24a}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial y^*} + \frac{H_0 - \sigma}{D(x,y)}\frac{\partial D}{\partial y}\frac{\partial}{\partial \sigma} \tag{24b}$$

$$\frac{\partial}{\partial z} = \frac{H_0}{D(x,y)}\frac{\partial}{\partial \sigma} \tag{24c}$$

Note: $\frac{H_0}{D(x,y)} \in [1, \infty >$. (If h(x,y) is negative somewhere then $\frac{H_0}{D(x,y)} \in \langle 0, \infty \rangle$).

The simplified continuity equation ($\nabla \cdot \vec{V} = 0$) then transforms according to:

$$\frac{\partial u}{\partial x^*} + \frac{(H_0 - \sigma)}{D}\frac{\partial D}{\partial x}\frac{\partial u}{\partial \sigma} + \frac{\partial v}{\partial y^*} + \frac{(H_0 - \sigma)}{D}\frac{\partial D}{\partial y}\frac{\partial v}{\partial \sigma} + \frac{H_0}{D}\frac{\partial w}{\partial \sigma} = 0 \tag{25}$$

The second term can be written as:

$$\frac{(H_0 - \sigma)}{D}\frac{\partial D}{\partial x}\frac{\partial u}{\partial \sigma} = \frac{\partial}{\partial \sigma}\left(\frac{(H_0 - \sigma)}{D}\frac{\partial D}{\partial x} u \right) + \frac{u}{D}\frac{\partial D}{\partial x}$$

and similary, for the fourth term:

$$\frac{(H_0 - \sigma)}{D}\frac{\partial D}{\partial y}\frac{\partial v}{\partial \sigma} = \frac{\partial}{\partial \sigma}\left(\frac{(H_0 - \sigma)}{D}\frac{\partial D}{\partial y} v \right) + \frac{v}{D}\frac{\partial D}{\partial y}$$

and for the last term of eq. (25):

$$\frac{H_0}{D}\frac{\partial w}{\partial \sigma} = \frac{\partial}{\partial \sigma}\left(\frac{H_0 w}{D}\right)$$

Utilizing this, eq. (25) can be written as:

$$\frac{\partial u}{\partial x^*} + \frac{u}{D}\frac{\partial D}{\partial x} + \frac{\partial v}{\partial y^*} + \frac{v}{D}\frac{\partial D}{\partial y} + \frac{\partial}{\partial \sigma}\left(\frac{H_0}{D} w + \frac{(H_0 - \sigma)}{D}\frac{\partial D}{\partial x} u + \frac{(H_0 - \sigma)}{D}\frac{\partial D}{\partial y} v\right) = 0 \qquad (26)$$

Now, defining :

$$\omega \equiv \frac{H_0}{D} w + \frac{(H_0 - \sigma)}{D}\frac{\partial D}{\partial x} u + \frac{(H_0 - \sigma)}{D}\frac{\partial D}{\partial y} v \qquad (27)$$

The transformed continuity equation can be written compactly as:

$$\frac{\partial (uD)}{\partial x^*} + \frac{\partial (vD)}{\partial y^*} + \frac{\partial (\omega D)}{\partial \sigma} = 0 \qquad (28)$$

By insertion it is easily verified that the new quantity ω in eq. (27) can be written as:

$$\omega = \vec{V} \cdot \nabla \sigma \qquad (29)$$

where $\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$ is the Cartesian wind velocity vector, and ∇ is the Cartesian gradient operator. Thus

$$\omega = \vec{V} \cdot \left(\frac{H_0}{D}\vec{k} + \frac{H_0 - \sigma}{D}\left(\frac{\partial D}{\partial x}\vec{i} + \frac{\partial D}{\partial y}\vec{j}\right)\right).$$

Since the vector $\left(\frac{H_0}{D}\vec{k} + \frac{H_0 - \sigma}{D}\left(\frac{\partial D}{\partial x}\vec{i} + \frac{\partial D}{\partial y}\vec{j}\right)\right)$ is a gradient vector of σ, it is directed normal to any of the model surfaces, i.e. $\sigma = \frac{H_0}{D}(z - h(x,y)) = \text{const.}$, pointing upwards. The upward direction is evident since the vertical component of this vector is always positive.

By dividing eq. (29) by the length of the gradient vector we define a new quantity, $\dot{\sigma}$, given by

$$\dot{\sigma} \equiv \frac{\omega}{\left\| \frac{H_0}{D}\vec{k} + \frac{H_0 - \sigma}{D}\left(\frac{\partial D}{\partial x}\vec{i} + \frac{\partial D}{\partial y}\vec{j}\right)\right\|} = \vec{V} \cdot \vec{n}_{\,z = \frac{D(x,y)}{H_0}\sigma + h(x,y)} \qquad (30)$$

which define the component of the wind velocity normal to the model surfaces, i.e. surfaces with constant values of σ. Positive values of $\dot{\sigma}$ indicate an upward directed wind.

Transformation of the different terms in eq. (4) now gives:

$$\frac{\partial c_i}{\partial t} \Rightarrow \frac{\partial c_i}{\partial t} \tag{31}$$

$$\frac{\partial}{\partial x}(uc_i) \Rightarrow \frac{\partial}{\partial x^*}(uc_i) + \frac{H_0 - \sigma}{D}\frac{\partial D}{\partial x}\frac{\partial (uc_i)}{\partial \sigma} \tag{32}$$

$$\frac{\partial}{\partial y}(vc_i) \Rightarrow \frac{\partial}{\partial y^*}(vc_i) + \frac{H_0 - \sigma}{D}\frac{\partial D}{\partial y}\frac{\partial (vc_i)}{\partial \sigma} \tag{33}$$

$$\frac{\partial}{\partial z}(wc_i) \Rightarrow \frac{H_0}{D}\frac{\partial}{\partial \sigma}(wc_i) \tag{34}$$

$$\begin{gathered} \frac{\partial}{\partial x}\left(K^{(H)}\frac{\partial c_i}{\partial x}\right) \\ \Downarrow \\ \frac{\partial}{\partial x^*}\left(K^{(H)}\left(\frac{\partial c_i}{\partial x^*} + \frac{H_0 - \sigma}{D}\frac{\partial D}{\partial x}\frac{\partial c_i}{\partial \sigma}\right)\right) + \frac{H_0 - \sigma}{D}\frac{\partial D}{\partial x}\frac{\partial}{\partial \sigma}\left(K^{(H)}\left(\frac{\partial c_i}{\partial x^*} + \frac{H_0 - \sigma}{D}\frac{\partial D}{\partial x}\frac{\partial c_i}{\partial \sigma}\right)\right) \end{gathered} \tag{35}$$

$$\begin{gathered} \frac{\partial}{\partial y}\left(K^{(H)}\frac{\partial c_i}{\partial y}\right) \\ \Downarrow \\ \frac{\partial}{\partial y^*}\left(K^{(H)}\left(\frac{\partial c_i}{\partial y^*} + \frac{H_0 - \sigma}{D}\frac{\partial D}{\partial y}\frac{\partial c_i}{\partial \sigma}\right)\right) + \frac{H_0 - \sigma}{D}\frac{\partial D}{\partial y}\frac{\partial}{\partial \sigma}\left(K^{(H)}\left(\frac{\partial c_i}{\partial y^*} + \frac{H_0 - \sigma}{D}\frac{\partial D}{\partial y}\frac{\partial c_i}{\partial \sigma}\right)\right) \end{gathered} \tag{36}$$

$$\frac{\partial}{\partial z}\left(K^{(Z)}\frac{\partial c_i}{\partial z}\right) \Rightarrow \frac{H_0}{D}\frac{\partial}{\partial \sigma}\left(K^{(Z)}\frac{H_0}{D}\frac{\partial c_i}{\partial \sigma}\right) = \left(\frac{H_0}{D}\right)^2 \frac{\partial}{\partial \sigma}\left(K^{(Z)}\frac{\partial c_i}{\partial \sigma}\right) \tag{37}$$

This means that the left hand side of eq. (4) (i.e. the time derivative and the advection terms) by the use of the expressions (31) – (34), can be expressed in the following way:

$$\frac{\partial c_i}{\partial t} + \frac{\partial (uc_i)}{\partial x^*} + \frac{H_0 - \sigma}{D}\frac{\partial D}{\partial x}\frac{\partial (uc_i)}{\partial \sigma} + \frac{\partial (vc_i)}{\partial y^*} + \frac{H_0 - \sigma}{D}\frac{\partial D}{\partial y}\frac{\partial (vc_i)}{\partial \sigma} + \frac{H_0}{D}\frac{\partial (wc_i)}{\partial \sigma}$$

$$= \frac{\partial c_i}{\partial t} + \frac{\partial (uc_i)}{\partial x^*} + \frac{\partial (vc_i)}{\partial y^*} + \frac{H_0}{D}\frac{\partial (wc_i)}{\partial \sigma} + \frac{\partial}{\partial \sigma}\left(\frac{H_0 - \sigma}{D}\frac{\partial D}{\partial x}uc_i\right) - uc_i\frac{\partial}{\partial \sigma}\left(\frac{H_0 - \sigma}{D}\frac{\partial D}{\partial x}\right)$$

$$+ \frac{\partial}{\partial \sigma}\left(\frac{H_0 - \sigma}{D}\frac{\partial D}{\partial y}vc_i\right) - vc_i\frac{\partial}{\partial \sigma}\left(\frac{H_0 - \sigma}{D}\frac{\partial D}{\partial y}\right)$$

$$= \frac{\partial c_i}{\partial t} + \frac{\partial (uc_i)}{\partial x^*} + \frac{\partial (vc_i)}{\partial y^*} + \frac{uc_i}{D}\frac{\partial D}{\partial x} + \frac{vc_i}{D}\frac{\partial D}{\partial y} + \frac{\partial}{\partial \sigma}\left(\frac{H_0}{D}wc_i + \frac{H_0 - \sigma}{D}\frac{\partial D}{\partial x}uc_i + \frac{H_0 - \sigma}{D}\frac{\partial D}{\partial y}vc_i\right)$$

$$= \frac{\partial c_i}{\partial t} + \frac{\partial (uc_i)}{\partial x^*} + \frac{\partial (vc_i)}{\partial y^*} + \frac{uc_i}{D}\frac{\partial D}{\partial x} + \frac{vc_i}{D}\frac{\partial D}{\partial y} + \frac{\partial (\omega c_i)}{\partial \sigma}$$

$$= \frac{\partial c_i}{\partial t} + \frac{1}{D}\left(\frac{\partial (uc_i D)}{\partial x^*} + \frac{\partial (vc_i D)}{\partial y^*} + \frac{\partial (\omega c_i D)}{\partial \sigma}\right) \tag{38}$$

Where the definition of the transformed vertical velocity, ω, defined by eq. (27), has been applied.

By the use of the expressions (35), (36) and (37) the diffusions terms in eq. (4) can be expressed as:

$$\frac{\partial}{\partial x^*}\left(K^{(H)}\left(\frac{\partial c_i}{\partial x^*}+\frac{H_0-\sigma}{D}\frac{\partial D}{\partial x}\frac{\partial c_i}{\partial \sigma}\right)\right)+\frac{H_0-\sigma}{D}\frac{\partial D}{\partial x}\frac{\partial}{\partial \sigma}\left(K^{(H)}\left(\frac{\partial c_i}{\partial x^*}+\frac{H_0-\sigma}{D}\frac{\partial D}{\partial x}\frac{\partial c_i}{\partial \sigma}\right)\right)+$$

$$\frac{\partial}{\partial y^*}\left(K^{(H)}\left(\frac{\partial c_i}{\partial y^*}+\frac{H_0-\sigma}{D}\frac{\partial D}{\partial y}\frac{\partial c_i}{\partial \sigma}\right)\right)+\frac{H_0-\sigma}{D}\frac{\partial D}{\partial y}\frac{\partial}{\partial \sigma}\left(K^{(H)}\left(\frac{\partial c_i}{\partial y^*}+\frac{H_0-\sigma}{D}\frac{\partial D}{\partial y}\frac{\partial c_i}{\partial \sigma}\right)\right)+$$

$$\left(\frac{H_0}{D}\right)^2\frac{\partial}{\partial \sigma}\left(K^{(Z)}\frac{\partial c_i}{\partial \sigma}\right) \qquad (39)$$

If, as in the original transformed version, all the additional terms from the horizontal turbulent diffusion in expression (39) are neglected, the transformed version of eq. (4) becomes:

$$\frac{\partial c_i}{\partial t}+\frac{1}{D}\left(\frac{\partial(uc_iD)}{\partial x^*}+\frac{\partial(vc_iD)}{\partial y^*}+\frac{\partial(\omega c_iD)}{\partial \sigma}\right)=\frac{\partial}{\partial x^*}\left(K^{(H)}\frac{\partial c_i}{\partial x^*}\right)+\frac{\partial}{\partial y^*}\left(K^{(H)}\frac{\partial c_i}{\partial y^*}\right)+$$
$$\left(\frac{H_0}{D}\right)^2\frac{\partial}{\partial \sigma}\left(K^{(Z)}\frac{\partial c_i}{\partial \sigma}\right)+R_i-S_i \qquad (40)$$

Where the applied wind velocity components obey eq. (27), i.e.:

$$\omega\equiv\frac{H_0}{D}w+(H_0-\sigma)\frac{u}{D}\frac{\partial D}{\partial x}+(H_0-\sigma)\frac{v}{D}\frac{\partial D}{\partial y}$$

or alternatively:

$$\omega\equiv\frac{H_0}{D}w-(H_0-\sigma)\frac{u}{D}\frac{\partial h}{\partial x}-(H_0-\sigma)\frac{v}{D}\frac{\partial h}{\partial y}.$$

4 Implementation of the new σ-coordinate transform in the EPISODE program code

From: $\sigma=z-h(x,y)$ to $\sigma=\frac{H_0}{H_0-h(x,y)}(z-h(x,y))=\frac{H_0}{D(x,y)}(z-h(x,y))$

The site-declaration must include the new site specific (and user defined) model height, H_0, and the 2D topography field, $h(x,y)$. The total model height, H_0, is calculated as the sum of the user defined thicknesses (in metres) of all of the σ-layers. These thickness values are either specified interactively in an AirQUIS window, or read from an input-file. Similarly, the 2D topography field, $h(x,y)$, is also specified by the user (AirQUIS window or file).

The following changes has been made: (Marked as: C_SIGMA... in the code)

Vertical velocity is calculated according to eq.(28)from the requirement of a divergence-free wind field.
See: ***src/mete/calcw.for*** and ***src/mete/cdive.for***

Vertical diffusion: The vertical diffusivity $K^{(z)}$ is multiplied by the square of the stretching factor: $(H_0/D(x,y))^2$ in: ***src/mete/cdzdt.for***.

Horizontal advection: Multiplies the concentrations with D(x,y) in: ***src/grid/advb4p.for*** and ***src/grid/advb4m.for***. The concentration values are then divided by D(x,y) at the end of these routines.

The stretching factor $H_0/D(x,y)$, which enters the advection/diffusion equation through the vertical turbulent diffusion terms, will normally vary between 1.0 and ∞. However, with large values of the stretch factor, the computation time increase severely. This is related to the stability requirements of the explicit numerical method applied for the vertical diffusion. This problem can be reduced by choosing H_0 large enough compared to the variations in the topography. Tests have revealed that with a maximum value of $H_0/D(x,y)$ = 1.2 no significant changes is found in computation time, as compared to a flat terrain simulation, while a value of 3.5 leads to a dramatic increase.

5 References

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ABSTRACT
The terrain following vertical coordinate previously applied in the Eulerian grid model EPISODE has been changed. In this report a detailed mathematical description of both the old and the new coordinate system is presented. In addition the necessary code changes for going from the old to the new system is described.

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Beskrivelse av Episodes sigma-koordinat transformasjon. Gammel og ny versjon.

KEYWORDS		
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ABSTRACT (in Norwegian)
Den terrengfølgende vertikalkoordinaten i den Eulerske spredningsmodellen EPISODE er blitt endret. I denne rapporten gis en detaljert matematisk beskrivelse av både det gamle og det nye koordinat-systemet. I tillegg beskrives kode-endringene som er nødvendige for å gå fra det gamle til det nye systemet.

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