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# ON OPTIMAL METEOROLOGICAL PREDICTION OF TRAJECTORIES THROUGH STOCHASTIC FLOW FIELDS

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SUMMARY

The dependence of some physical processes upon environmental flow and flow prediction is approached. Procedures for obtaining sufficiently accurate prediction methods for these processes are outlined.

# ON OPTIMAL METEOROLOGICAL PREDICTION OF TRAJECTORIES THROUGH STOCHASTIC FLOW FIELDS

## 1 INTRODUCTION

When someone request a prediction of atmospheric (geophysical) flows, for the purpose of optimal control of a physical process or system, the forecaster will normally respond with one of the two alternative prediction methods: The most sophisticated available or the simplest and first that comes to his mind. Neither method may be "cost effective" for the user. It is not even given that the prediction will improve significantly the operation of the users process.

We will discuss a simple approach towards estimating what are sufficiently accurate prediction methods for a class of physical processes. The processes are associated with trajectories of (ensembles of) particles or signals through geophysical fields, exemplified with the following subclasses.

a): <u>"Accidential pollution"</u> Here an ensemble of (unfavourable) particles has been generated at a more or less precisely known location. The problem may be to estimate or predict geophysical flows so as to be able to extremize the effort to compensate for the effect of the polluting cloud (optimal distribution of warnings or equipment).

b): <u>"Predicted fire"</u> When first round "fire" accuracy is desired, the atmospheric fields are in principle measured and predicted before the fire so that the launch conditions is appropriate for the actual atmospheric conditions. The object fired may be a ball, projectile, bomb, rocket or it may be the generation of a cloud of some material. In these processes a meteorological prediction is required so that the initial conditions for the trajectories can be chosen so as to extremize the effect sought.

c): "Ray tracing" In this subclass the problem is to determine the source of a signal that has passed through the atmosphere and is received at some location. It is required to estimate the "refractive index" in such a manner that optimal source location accuracy is achieved. The signal may be sound or electromagnetic waves and also a cloud.

Traditionally, each subclass is accosiated with different communities of scientists. Each has developed accurate, but complicated models to describe the aspects of most traditional interest. The prediction aspect is common to all, and it is significant that it, to some degree, may be discussed in a general, simple manner.

## 2 STOCHASTIC COMPENSATION

Although we may control a physical process by means of initial conditions and measure its "effectiveness" at the end point of trajectories only, the problem is formulated along the lines proposed by system control theory. The most relevant concepts from this theory are covered by for instance Wiener (1), Pontryagin et al. (2), Meditch (3) and Box and Jenkins (4). An instructive review is given by Athans (5).

### 2.1 Problem statement

The trajectory of a particle is described by its position and velocity vector, $x(\tau)$. The atmospheric (geophysical) fields along the trajectory is denoted by $u(x(\tau))$. The equation of motion is assumed to be of the form

$$\left.\begin{aligned} \frac{dx(\tau)}{d\tau} &= f[x(\tau),\ u(x(\tau))] \\ x(t_o) &= \text{initial conditions} \end{aligned}\right\} \qquad (2.1)$$

Here f may be a nonliear vector-valued function. $\tau$ is a time coordinate $t_o$ and t are initial and "final" time coordinates, respectively. It is assumed that for a given u, a unique solution x(t) exists which satisfies the initial conditions. In system control terminology x is the state variable and u is the disturbance. In the classical system control problems there is also a control vector. Our only possibility of controlling or compensate the trajectory is to vary the initial condition, $x(t_o)$. The performance of the process is measured with a scalar valued index $\nu$. The problem can be loosely formulated as follows: Estimate or predict atmospheric flow fields $u(\tau)$ so that a performance index $\nu$ is extremized.

## 2.2 Separation of the problem

System control theory provides a systematic and simple approach to the problem formulated above. Conditional to weak restrictions on the system and disturbance structure, it may be separated into the following simpler problems.

### 2.2.1 Reference state

Suppose that all the variables in Equation (2.1) were deterministic and known. Then this equation can be solved so that for each $x(t_o)$ and $u(\tau)$ there is one trajectory $x(\tau)$, $\tau > t_o$. Given $u(\tau)$ one may compute the particular $x(t_o)$ that makes x(t) a desired target value, or from known x(t) at some time, to track the signal backwards. A trajectory obtained with "standard" u(t) is called a reference state or trajectory. In the accident class of problems, a), the reference trajectory may be chosen as the predicted, a purposely simple one.

In the classical system control problem where there are control possibilities along the trajectory, the control that will bring the system along a reference trajectory is found by variational analysis, or if the admissible region of the control is limited, by Pontryagins maximum principle.

### 2.2.2 Deterministic controller

The Equation (2.1) is normally nonlinear and complicated to solve. This is a sufficient reason for seeking a simpler approach. Often can small changes in atmospheric fields change trajectories only little, and as atmospheric field usually vary most over larger distances, the following hypothesis are stated:

H1a: linearized equations are valid
H1b: atmospheric fields along the actual and reference trajectory are the same.

When the pertubations from the reference state are denoted the same as the unpertubed variables, linearization of Equation (2.1) give.

$$\left.\begin{aligned} \frac{dx(\tau)}{d\tau} &= \frac{\partial f}{\partial x} x(\tau) + \frac{\partial f}{\partial u} u(\tau) \\ x(t_o) &= \text{initial deviations.} \end{aligned}\right\} \qquad (2.2)$$

The solution is

$$x(t) = A(t,t_0)x(t_0) + \int_{t_o}^{t} A(t,\eta) \frac{\partial f}{\partial u} u(\eta)d\eta \qquad (2.3a)$$

Here t and $\eta$ are time or length coordinates along the reference trajectory. $A(t,\eta)$ is the (known) state transision matrix. It is convenient to write Equation (2.3a) formally as

$$x(L) = \frac{\partial x(L)}{\partial x(0)} x(0) + \frac{\partial x(L)}{\partial u} \bar{u}^L \qquad (2.3b)$$

with

$$\left.\begin{aligned} \bar{u}^L &= \int_0^L \kappa(L,\eta)\; u(\eta)d\eta \\ 1 &= \int_0^L \kappa(L,\eta)d\eta \end{aligned}\right\} \qquad (2.4)$$

Under the hypothesis H1, it is often convenient to consider L as the trajectory length. The matrices $\frac{\partial x(L)}{\partial x(0)}$, $\frac{\partial x(L)}{\partial u}$ and $\kappa(L,\eta)$ are in principle known. A typical ballistic weight function $\kappa(L,\eta)$ for artillery projectiles is shown in Figure 2.1.

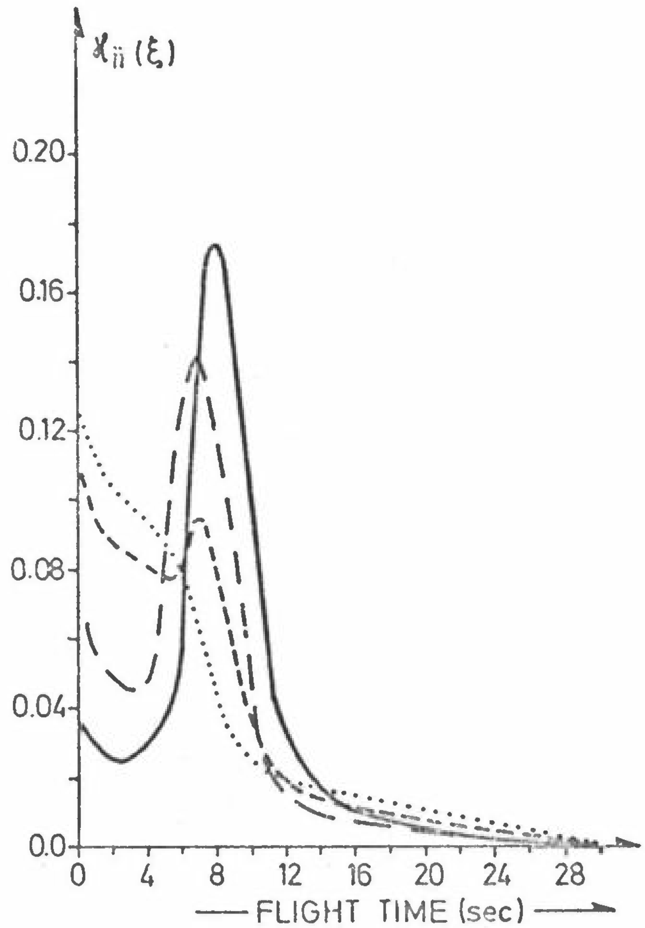


*Figure 2.1: Components of the Ballistic weight Function for an Artillery Reference Trajectory. Maxima in the neighbourhood of passage from supersonic to subsonic velocity.*

Roughly, the hypothesis H1 implies that the particle is heavy and fast compared to air particles. In the other extreme

H2: a particle follows the atmospheric flow.

This idealisation do also enable a formally simple equation for the spatial coordinate of the particle

$$x(t) = x(t_o) + \int_{t_o}^{t} u(x(\tau))d\tau \qquad (2.3c)$$

In Equation (2.3a) or (2.3b) the flow variables are Eulerian and in (2.3c) they are Lagrangian. Lumley (6) has discussed under what conditions small particles experience appoximately an Eulerian field or a Lagrangian. Properties of functionals like (2.3a) and (2.3c) have been extensively discussed in the litterature. Unless explicitely stated, we will not distinguish between the two types. Formally the relation between $x(t_o)$, $u(\tau)$ and $x(t)$ may, in any of the two cases, be written as in Equation (2.3a) or (2.3b).

In the subclass of problems where initial condition compensation is required for target hit, the (deterministic minimum quadratic) compensator is obtained from Equation (2.3) by setting $x(L) = 0$. This gives

$$x(0) = -\left(\frac{\partial x(L)}{\partial x(0)}\right)^{-1} \frac{\partial x}{\partial u} \; \bar{u}^{L} \qquad (2.5)$$

If the deviation, $u(\tau)$ from a reference atmosphere were known prior to $t_o$, Equation (2.5) would provide the appropriate adjustment of the initial condition, $x(0)$. In the accident class of problems Equation (2.3) would enable calculation of the deviation from some predicted, purposely simple trajectory. In the ray-tracing class of problems the source is located at

$$x(t_o) = A^{-1}(t,t_o)\left[x(t) - \int_{t_o}^{t} A(t,\eta)\,\frac{\partial f}{\partial u}\,u(\eta)d\eta\right] \qquad (2.6)$$

so that if $u(\tau)$; $t_o < \tau < t$ and $x(t)$ at some time, $t$, were known, the trajectory along which $x(t_o)$, $t_o < t$ were located could be computed.

An example where neither of the hypothesis H1 or H2 are applicable is for modelling of the trajectory of a nonguided model (paper) plane. It may be fast and heavy in relation to its surroundings, but it may still change its trajectory radically due to small changes of the surrounding flow. The use of these simple equations must therefore be justified in each problem to be analyzed.

### 2.2.3 Stochastic compensation

In reality the geophysical fields are not deterministic and future values are not known. The separation principle provides the necessary modification to this situation. It states that the optimal stochastic control is achieved by combining the best predictor and the deterministic control. When the best predictor, $\hat{u}$, is introduced into the control Equations (2.5) and (2.6) the system deviation from the target values is, for the first two subclasses of problems, (a and b);

$$x(L) = \frac{\partial x(L)}{\partial x(0)}\left[x(0) - x(0)\right] + \frac{\partial x(L)}{\partial u}\left[\bar{u}^L - \hat{\bar{u}}^L\right] \qquad (2.7a)$$

and for the tracing problem, (c):

$$x(0) = -\left[\frac{\partial x(L)}{\partial x(0)}\right]^{-1} \frac{\partial x(L)}{\partial u}\left[\bar{u}^L - \hat{\bar{u}}^L\right] \qquad (2.7b)$$

Sometimes it is convenient to imagine that x(L) and x(0) are realized as continuous functions of time. In the second subclass it is then imagined that a continuous flow of projectiles (rockets, bombs) are launced along the same reference trajectory like water from a garden house.

## 2.3 Generalization to ensembles of particles

Suppose that two identical heavy and fast particles are released with identical initial conditions, one at time $t_i$, and the other at time $t_j$. The difference in impact coordinates is

$$x(t_j;L) - x(t_j;L) = \int_0^L \kappa(L,\eta) \left[ u(\eta;t_i) - u(\eta;t_j) \right] d\eta \qquad (2.8)$$

Except for rockets, the weight function $\kappa(L,\eta)$ will vary smoothly with $\eta$. Its support is approximately L/2. The integral will not change much before u has changed coherently over the support of $\kappa(L,\eta)$. The size of atmospheric eddies that can contribute to a coherent change are larger than approximately

$$\Delta r^f \approx L/2 \qquad (2.9)$$

With a mean wind, U, it takes a characteristic time

$$\Delta t^f \approx L/2U \qquad (2.10)$$

before the smallest of these eddies have changed significantly over the support of $\kappa(L,\eta)$. The eddies that can contribute to the difference $u(\eta,t_i) - u(\eta,t_i)$ have timescale not much larger than $|t_i - t_j|$. For particles released such that $|t_i - t_j| < \Delta t^f$, the integral above will therefore be small. All these particles will therefore have approximately the same x.

If two particles are released simultaneously along approximately paralell reference trajectories from locations $r_i$ and $r_j$, analogeous arguments apply. All particles released such that $|r_i - r_j| < \Delta r^f$ have approximately the same x.

In the case of light particles the hypotesis H1b is not valid. The size of the ensemble grow with time because the Lagrangian velocity will vary from particle to particle "regardless" of the initial size of the ensemble (turbulent diffusion). However, the functional (2.3b) is linear so that the motion of the center of gravity of the ensemble is described by the same kind of equation. x may thus also in this case be understood to mean the coordinates of the center of gravity of the ensemble. The smallest length and time scales that can contribute significantly to different center of gravity motion is as small as:

$$\Delta r^f \simeq \sigma_b, \quad \Delta t^f \simeq \sigma_b/u_* \qquad (2.12)$$

Here $\sigma_b$ characterizes the size of the ensemble and $u_*$ is a characteristic velocity of the small scale fluctuations. However, even in this case it is the eddies of larger dimensions than the size of the trajectory, $(L, t-t_o)$, that can contribute most to the spatial location of the cloud.

## 2.4 Geometry

In the description so far, the spatial and time geometry of the problem has only been implicitly mentioned The deviation, x, should be assigned to a referenc (or predicted) trajectory in physical space and time. The predicted atmospheric field, $\hat{u}$, should be assigned to the same trajectory and also to the geometry associated with the measurement procedure. Faced with this, we make two idealizations which usefulness must be justified.

H3) All trajectories are the same within a resolution characterizized by $(\Delta r^f, \Delta t^f)$.
H4) All trajectories involved are of small length and time dimensions compared to other length and time scales in the problem.

When the idealization H3 is used, we need not explicitly refer to trajectories, and when the idealization H4 is used, the trajectories are represented as points in physical space and time.

## 2.5 Scematic representation of the process

At this stage it is convenient to illustrate schematically the physical process under discussion (Figure 2.2).

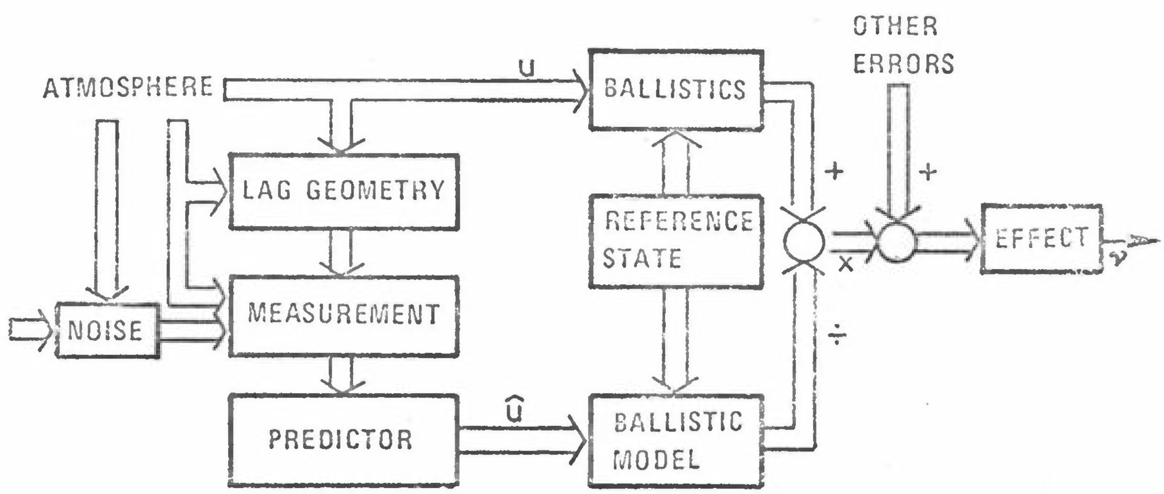


*Figure 2.2: Feedforward Control of Trajectory Accuracy.*
*$u$ - Atmospheric variables*
*$\hat{u}$ - Predicted values*
*$x$ - Miss distance after adjustment*
*$\nu$ - Accuracy norm*

The upper branch illustrates the actual effects of atmospheric variables on the trajectory. The lower branch illustrates how this is predicted. The difference between the two constitute the meteorological contribution to the "miss distance". Tradition in meteorology is to be interested and discuss extensively only the first stage of such a process, that is to produce a prediction, $\hat{u}$. However, it is the effect of the prediction error that enables the benefit of the meteorological effort to be judged.

# 3 SIMPLIFIED SMALL SCALE DESCRIPTION

## 3.1 Gross characteristics of atmospheric flows

Some general properties of atmospheric flows are useful when measurement and prediction methods are discussed. Atmospheric flows are four-dimensional, stochastic fields governed by nonlinear, conclosed equations. It is obvious that the tools available to discuss the majority of such flows are not sharp. We must therefore be satisfied with less than stringent arguments. To focus attention at the most relevant scales of motion to us, it is stated that the typical horizontal distances and times between measurement locations and the particle trajectory are assumed to be of the order

$$|r_o - r_1| \leq O(100 \text{ km})$$
$$|t_o - t_1| \leq O(6 \text{ hr}) \qquad (3.1)$$

The vertical extent of the trajectories are of the order

$$z \leq O(1 \text{ km}) \qquad (3.2)$$

Atmospheric fluctuations of scales larger than these characteristic distances are corrected for when compensating the trajectory for the prediction. With the particular choice above, it is smaller scales than the smallest synoptic scales of motion that can contribute to system error.

H5): Atmospheric variables usually have most energy associated with the larger scale fluctuations. The pressure is most dominantly red, but also horizontal velocity components and "passive" scalars are. The exception is the vertical velocity component.

H6): Atmospheric fields normally vary much more rapidly along the vertical than along horizontal coordinates.

### 3.1.1 Flow field variables

The meteorological idealization H6 enables the quasistatic approximation to be used, as for instance discussed by Eliassen (7). From vertical profiles of temperature and moisture and surface pressure, the profiles of pressure and density may be computed by means of the hydrostatic equation. To be spesific the variables that suffice to characterize the atmosphere for our purpose are thus:

| | |
|---|---|
| Velocity components | $(u, v, w) = (u_1, u_2, u_3)$ |
| Temperature | $T = u_4$ |
| Pressure at ground level | $p_o = u_5$ |
| Moisture | $\rho_v = u_6$ |

The refractive index for electromagnetic and sound waves are known when these variables are given, as indicated by for instance Tatarskii (8). Although the above variables are related through the equations of motion, they may probably be considered stochastic independent for most practical problems.

## 3.2 Ballistic model simplifications

In functionals of the kind

$$\bar{u}^L = \int_o^L \kappa(L,\eta)u(\eta)d\eta \tag{3.4}$$

it is the details of the normalized weight function $\kappa(L,\eta)$, the filtering of small scale variations, that causes analytical complexity. Besides being of complex shape for one reference trajectory for one kind of particle, it may vary considerably from one reference trajectory to another and from one kind of particle to another. Unless otherwise stated, it is assumed that the ballistic operator used, $-L$ , is the simplest or most convinient one to give a small enough error $R_f^2$.

$$\left(\frac{\partial x}{\partial u}\right)^2 E \left[ \int_0^L \kappa(L,\eta)u(\eta)d\eta - \bar{u}^L \right]^2 \leq R_f^2 \tag{3.4}$$

As atmospheric fluctuations have more energy at larger scales than L and $\kappa(L,\eta)$ usually do not amplify the effect of small scale eddies, we may normally use a very simple operator $-L$ such as for instance

$$\bar{u}^L = \int_{\Delta r^f} u(\eta)d\eta \tag{3.5}$$

3.2.1 Predicted trajectory simplifications

With the meteorological idealisation H6, we may introduce a simpler expression for the predicted deviation

$$\bar{\hat{u}}^L = \int_0^L \kappa(L,\eta)\hat{u}(\eta)d\eta \tag{3.6}$$

The time and horizontal scale covered by the trajectory are characterized by $(t - t_o)$ or $\Delta r^f$. The vertical scale is Z. When Z is not small relative to $\Delta r^f$, the particle is usually heavy and fast so that the flight time $(t - t_o)$ is small relative to other time scales of relevance. The meteorological idealisation H6 may then be spesified to a statement involving that $\hat{u}(\tau)$ varies much more over Z than over $\Delta r^f$. The integral (3.6) may then be transformed into an integral along the height coordinate only.

$$\bar{\hat{u}}^L \simeq \int_0^Z \kappa(Z,z)\hat{u}(z)dz = \bar{\hat{u}}^Z \tag{3.7}$$

When Z is much less than $\Delta r^f \approx L/2$, the hypotesis H6 is not applicable as above. If then $\hat{u}$ is predicted constant over L it may as well be prediced constant over Z. That is

$$\bar{\hat{u}}^{Z} \simeq \hat{u} \qquad (3.8)$$

Also for the predicted trajectory it is assumed that the operator used, -Z, is a simple one satisfying an inequality of the type

$$\left(\frac{\partial x}{\partial u}\right)^2 E\left[\int_0^Z \kappa(Z,z)\hat{u}(z)dz - \bar{\hat{u}}^{Z}\right]^2 \leq R_p^{\,2} \qquad (3.9)$$

Again the details of $\kappa(Z,z)$ will normally not be of a shape that amplify the effect of atmospheric fluctuations smaller than Z. We may therefore normally use simple approximations for the operator $\bar{\hat{u}}^{Z}$ such as for instance integration over the support of $\kappa(Z,z)$: $\Delta Z^f$.

$$\bar{\hat{u}}^{Z} \simeq \int_{\Delta Z^f} \hat{u}(z)dz \qquad (3.10)$$

Both $R_f^{\,2}$ and $R_p^{\,2}$, and thus the approximations allowed, may vary with what aspect of the problem that is studied.

In some cases it may be useful to have an explicite expression for the difference between the operators -L and -Z. This is obtained by noticing that the operators are roughly low-pass filters with cut-off at approximately $1/\Delta r^f$ and $1/\Delta Z^f$ respectively. That is

$$\bar{u}^{Z} \simeq \bar{u}^{L} + u^{T} \qquad (3.11)$$

$$E(u^T)^2 \simeq \int_{1/\Delta r^f}^{1/\Delta r^f} \phi(k)dk \qquad (3.12)$$

### 3.3 The response of measurement systems to atmospheric fluctuations

A measured atmospheric variable $u^m$ is the output variable of a system that responds to the atmospheric field u. There exist an extensive meteorological literature on the response characteristica of different sensors. Propellers and wanes have for instance been discussed by Mc Cready (9). The response of balloons and parachutes have been discussed by Fichtl (10,11) and Mc Cready (12). Remote sensing techniques have been reviewed by Kjelaas (13).

The measurement system that can be of any use to compensate out systems are low pass systems. That is: atmospheric variations of scales larger than some lower limit, the response distance, are faithfully recorded. In our problems the responce distance is usually much smaller than other length scales. Some measurement systems are contaminated by self-induced fluctuations. Self-induced fluctuations are definitely non-desired and it is assumed they may cheaply be avoided. For our purpose, the measurement methods are therefore characterized by the measurement error $\varepsilon$ only, so that

$$u^m = u + \varepsilon \tag{3.13}$$

### 3.4 Local prediction method

A meteorological measurement is commonly accepted as meaningful without hesitation. However, this is only so if significant information about a flow field can be deduced from it. It is reasonable to believe that there are an infinite number of possible local flows satisfying the constrains set by a measurement. In the language of information theory (Shannon and Weaver 14) it is reasonable to believe that one measurement only convey limited information about a certain flow. The idealization H6 enables increased (apparent?) value of a measurement.

A consequence of this idealization is that meteorologists almost always try to measure and predict vertical profiles of atmospheric variables. It is understood that the fields are predicted constant in a horizontal area around the measurement site. From a measurement at one point or along (balloon) trajectory, information is thus obtained about the flow in a relative large volume of physical space.

### 3.4.1 Geometric idealization

In the case when Lagrangian or quasi-Lagrangian measurement techniques are applied, a geometric idealisation analogous to H4 have to be used to avoid the analytic difficulties associated with assigning the measured value to a trajectory The trajectory used for the measurement is characterized by a time $\Delta t^m$ and height $Z^m$ as illustrated in Figure 3.1. The horizontal distance covered is given when the mean wind U is. These variables enables estimates of the interval where the local prediction method must be used: Atmosperic fields are predicted constant, $\hat{u} = u^m$, at least over times $\Delta t^m$ and horizontal distances $\Delta r^m = U\Delta t^m$. The measured variables are assigned to the "center of gravity" of the trajectory $(r_1, t_1)$. If the measurement height $Z^m$ is higher than Z, the vertical component of the vector $r_1$ is set equal to Z. If $Z^m < Z$, it is set equal to $Z^m$. This geometrical idealisation is consistent with the use of the local prediction method over scales $\Delta t^m$ and $\Delta r^m$.

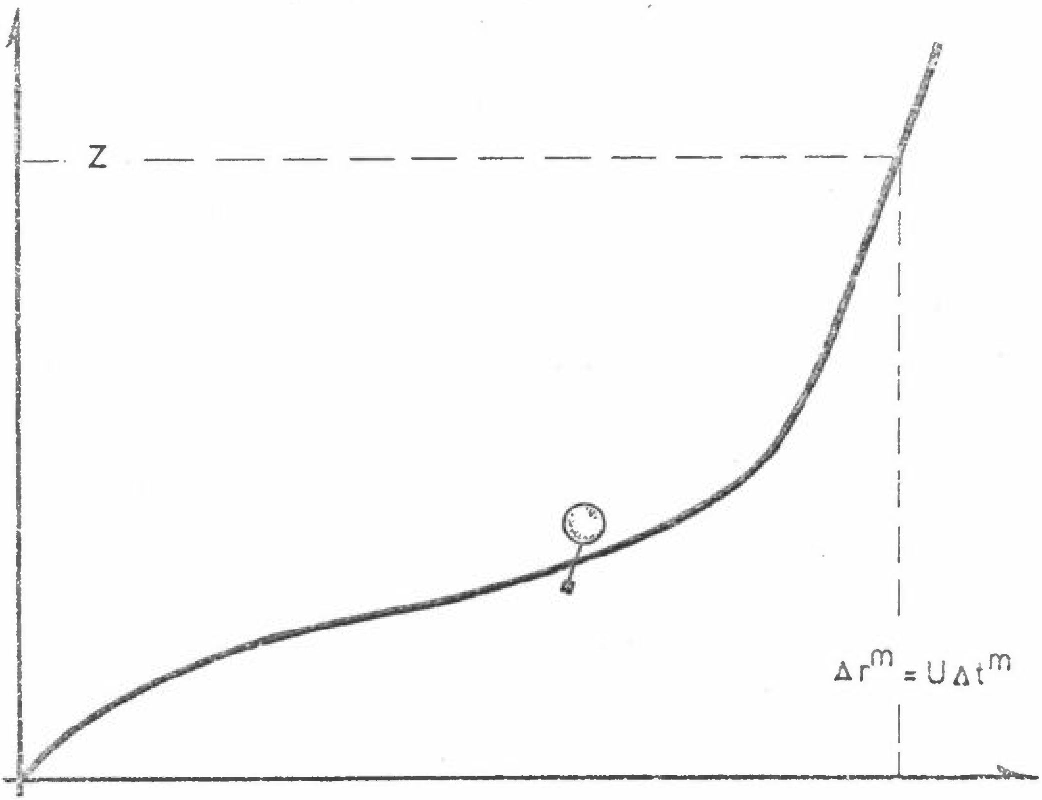


*Figure 3.1. Ballon trajectory in a vertical plane parallel to the mean wind vector.*

An approximate expression for the system error caused by this idealisation is estimated by

$$\frac{\partial x}{\partial u}\,[u(r_1 + \Delta r^m, t_1 + \Delta t^m) - u(r_1, t_1)] \qquad (3.14)$$

with variance

$$R_m^2 = \left(\frac{\partial x}{\partial u}\right)^2 E\,[u(r_1 + \Delta r^m, t_1 + \Delta t^m) - u(r_1, t_1)]^2$$

$$\simeq \left(\frac{\partial x}{\partial u}\right)^2 D_{11}(\Delta r^m, \Delta t^m)$$

$$\simeq \left(\frac{\partial x}{\partial u}\right)^2 2 \int_{1/\Delta r^m}^{\infty} \Phi_{11}(k)\,dk \qquad (3.15)$$

Here D is the structure function and $\Phi$ is the spectrum of u. When the error given by Equation (3.15) is small enough, the geometric idealization H4 may be used in the meaning referred to above.

The vertical resolution of a measured profile need not be better than to resolve variation that are stationary over times $\Delta t^m$ and horizontal distances $\Delta r^m$. In fully developed turbulence this scale is of the order $\Delta r^m$. In stable stratified flows it is normally smaller, as required by H6.

### 3.4.2 Vertical velocity is predicted zero

H7): The vertical velocity has most energy at isotropic scales of motion, smaller than the height above the ground.

Therefore it is almost impossible to obtain a measurement of this variable that is representative over large enough horizontal distances. If the vertical velocity measurement is to be of any value it must be done in the immediate neighbourhood of the system trajectory. This is, with present day technology, associated with unacceptably high cost. It is therefore decided not to measure vertical velocity. Instead it is always predicted to be zero. With this prediction the system variance contribution from the vertical velocity is approximately

$$R_W{}^2 \simeq \left(\frac{\partial x}{\partial u_3}\right)^2 \int_0^{1/\Delta r^f} \Phi_{33}(k)\,dk \qquad (3.16)$$

## 3.5 Small scale response

It has been indicated that a detailed discussion of the system responce to the smallest scales of atmospheric fluctuations may be complicated. These fluctuations are the least important in causing system error. In some cases their effects need therefore only to be roughly estimated.

The effects of the larger scales are more simply described when the time and spatial scales are much larger than

$$\Delta t \approx \max \begin{cases} t-t_o \\ \Delta t^f \\ \Delta t^m \end{cases}$$

$$\Delta r \approx \max \begin{cases} \Delta r^f \\ \Delta r^m \end{cases} \qquad (3.17)$$

$$\Delta z \approx \Delta Z^f$$

We may then predict the fields constant over ($\Delta r$, $\Delta z$, $\Delta t$) and assign $\bar{u}^L$ to a coordinate ($r_o$, $t_o$) and $u^m$ to a coordinate ($r_1$, $t_1$). In the equation

$$x(L) \propto \frac{\partial x}{\partial u} [\bar{u}^L - \hat{u}^{-Z}] \qquad (3.18)$$

it may then be allowed to use very simple expressions for the operators -L and -Z. The most essential aspects of the trajectories is then parametrizized by the coordinates ($r_i$, $t_i$) and weight functions ($\partial x/\partial u$).

When $R_w^2$ is larger than $R_m^2$, $R_f^2$ and $R_p^2$ the approximations above are as accurate as they need to be. When this is not the case, the largest of these error will be a measure of the model accuracy. The model is appropriate if this error causes acceptable process performance.

## 4 A SIMPLE ACCURACY NORM

When process performance or quality is to be measured, there will be difficulties in agreeing upon its proper measure. The "utility" or "loss" of decision theory, Glahn (15), "entropy" of information theory and "performance index" or "cost function" of system control theory are alle disputable quality measures. Contrary to the norms normally used in mathematics, these must primarily be chosen so as to have relevance to a particular system performance, not primarily to simplify the analysis.

In system control it is traditional to use a performance index that is quadratic in the deviation, x, from some target trajectory in phase space. It is simple, and making it small ensures that linearized system equations are valid (5). We will suggest that a simple quadratic measure also is of relevance for some of the problems we have set out to analyze. The cost would in our case be associated with obtaining the meteorological prediction. A very accurate prediction would imply a small deviation vector but possibly also a high cost. Such a prediction might therefore not be optimal. As the uncertainties in associating cost to a prediction method is very high, we will not discuss cost in this report, only bear in mind that it matters.

The interest is restricted to processes where several, $N \geq O$ (10), particles are released simultaneously along the same reference trajectory (approximately). Either because of purpose or randomness, the individual trajectories will be different. The dimension of the particle cloud along the coordinate j is denoted by $s_j$. When the differential coordinate of particle i is denoted $x_j^i$, and the differential coordinate of the senter of gravity of the ensemble is $x_j$, $s_j$ could be defined as the distance to a certain concentration of particles or as

$$s_j = \frac{1}{N} \sum_{i=1}^{N} (x_j^i - x_j)^2 \qquad (4.1)$$

When N is large enough it seems reasonable that there is a large probability of effect or danger inside the area or volume characterized by $s_j$; j=1,2,...M, while outside, the probability of effect is small. This condition may be expressed with the positive scalar:

$$\nu^2 = \sum_{j=1}^{M} \frac{(x_j)^2}{(s_j)^2} \qquad (4.2)$$

as

$$\begin{aligned} \nu^2 &\leq O(1) \leftrightarrow \text{Effective} \\ \nu^2 &> O(1) \leftrightarrow \text{Ineffective} \end{aligned} \qquad (4.3)$$

While the condition $|\nu| \leq O(1)$ must be highly controversial as a condition for process effectiveness, it should be of some relevance for the allowable process accuracy. This provides a much more useful norm for process inaccuracy than just stating that it should be small.

## 4.1 Examples of applicability

### 4.1.1 Weapon delivery

With modern fire control equipment the only significant contribution to the mean point of impact error is the meteorological. The minimum size of a salvo is the ballistic dispersion.

The relevant dimension, M, is often two spatial coordinates only. Either it is in a plane normal to the firing range or it is the horizontal plane. The relevance of the simple measure (4.3) for the effectiveness of an artillery salvo has been discussed by Eidsvik (16). The generalization to other types of salvo firings is trivial.

### 4.1.2 Trajectory of a cloud

Let us now address ourselves to aspects of the theme; Turbulent dispersion of puffs in the atmosphere. In this area meteorologists have been concerned with accurate prediction of cloud size given the flow parameters. A measure like (4.2) may not seem relevant to the experts. However, we must recall that we are not primarily concerned with the prediction as such. The purpose is primarily to compensate or control a process in a nearly optimal manner.

Suppose that a puff of toxic material has been released. Then our problem may be, with limited resouces, to distribute warnings and equipment, in time, to the areas where the danger is most immediate. It seems reasonable that the prediction accuracy of the most dangerous locations should then be as accurate as

$$\max\ (\nu^2(t)) < O(1) \quad ; \qquad t < t_{max}$$

Another reason why this prediction accuracy seems to be desireable is the possibility of directing detection equipment to locations where positive detection is probable. Detection probability for non-remote sensors is high when $\nu^2 \leq O(1)$ and low when $\nu^2 > O(1)$. Knowing the trajectory up to a given time, seems to enable increased prediction accuracy for later times. (Feedback instead of feedforward).

It is obvious that the above class of performance measures is not always relevant. For instance would a performance index relevant to the subclass of problems where the source of a signal is sought, be more complicated. It would reasonably be associated with the purpose of detection and would thus involve other systems than those associated with the trajectory and prediction methods.

## 4.2 The approximate distribution of risk

The stochastic variable $\nu$ (risk function) has a distribution that is given when the distribution of the process variables are known. For the purpose of obtaining an approximate estimate of this distribution at a fixed time it is assumed that $x_j^i$ and $x_j^i - x_j$ are normally distributed and stochastic independent with mean values 0 and variances $\sigma_{xi}^2$ and $\sigma_{bi}^2$. Then

$$\frac{(x_j)^2}{\frac{1}{N}\Sigma(x_j^i - x_j)^2} \simeq \frac{\sigma_{xj}^2}{\sigma_{bj}^2} F(1,N) \qquad (4.4)$$

where F(1,N) is a F-distributed variable. The performance index is then distributed approximately like

$$\nu^2 = F(1,N) \sum_{j=1}^{M} \frac{E(x_j)^2}{(\sigma_{bj})^2} \; ; \qquad N \geq O(10) \qquad (4.5)$$

To obtain effect it is required that the probability of $|\nu| < 1$ be large, say

$$P(|\nu| < 1) > 0.7 \qquad (4.6)$$

As long as $N > O(10)$, P turns out not to be affected very much by N. The condition above is approximately equivalent to the simpler condition on the cost function (expected risk):

$$E\nu^2 \leq 1 \qquad (4.7)$$

That is

$$\sum_{j=1}^{M} \frac{E(x_j)^2}{(\sigma_{bj})^2} \leq 1 \qquad (4.8)$$

A relevant accuracy norm is therefore a quadratic measure with all convenient properties of such.

In the discussion above the time has been kept constant. When $\nu$ over a time interval might be of relevance and spatial components of x are the most essential, the "integral scales" of $\nu(t)$ is normally large or infinite.

### 4.2.1 Model accuracy requirements

It is now possible to formulate more precise requirements to the accuracy of modelling the process. The error of modelling should contribute to $E\nu^2$ with factors significantly smaller than one.

$$\frac{R_f^2}{\sigma_b^2} < O(1), \quad \frac{R_p^2}{\sigma_b^2} < O(1) \text{ and } \frac{R_m^2}{\sigma_b^2} < O(1) \qquad (4.9)$$

When these inequalities are fulfilled we have a sufficiently accurate model of the prosess.

## 5 NONLOCAL PREDICTION METHOD

Suppose now that the suggested simple description is accurate enough. Then a simple framework to discuss the error contribution from the most important, energetic atmospheric fluctuations of scales larger than $\Delta t$ and $\Delta r$ is available.

### 5.1 Candidate methods

For prediction over larger time and spatial distances than $\Delta t$ and $\Delta r$ we may have to use sophisticated prediction methods to obtain $|\nu| \leq 1$. If so many observations are available that accurate enough initial conditions for the equations of motion can be constructed, we may think of dynamic prediction methods being used. For some processes this may be the appropriate class of prediction methods. However, for the class of processes considered such methods would be too inaccurate if the synoptic network were used and too costly and time consuming if a denser network were used. A more realistic class of prediction methods for time and spatial scales

$$\begin{aligned} O(\Delta t) &< |t_o - t_1| < O(6\text{ hr}) \\ O(\Delta r) &< |r_o - r_1| < O(100\text{ km}) \end{aligned} \qquad (5.1)$$

would be linear prediction schemes:

$$\hat{u}(z;r_o,t_o) = \sum_{j,k} \phi_{jk} u^m(z;r_j,t_k) \qquad (5.2)$$

Here $u^m(z;r_j,t_k)$ is the measured vertical profile at the spatial location $r_j$ and time $t_k$. This class of prediction methods has been discussed by for instance Gandin (17). Box and Jenkins (4) and Akaike (18). $\phi_{jk}$ is chosen so as to minimize the prediction error of $u(z;r_o,t_o)$. In principle $\phi_{ij}$ is known when the stochastic structure of $u^m$ and the localities $(r_i;t_j)$ are given. Here we will only give a rough outline of how this class of prediction methods enters formally in the description of our problem. The predicted variable of Equation (3.17) is

$$\hat{\bar{u}}^Z(r_o,t_o) = \int_o^Z \kappa(L,\tau) \sum_{jk} \phi_{jk} u^m(z;r_j,t_k)dz \qquad (5.3)$$

Another alternative (not more accurate) predictior of process output would be proportional to

$$\begin{aligned} \hat{\bar{u}}^Z(r_o,t_o) &= \sum_{jk} \phi^*_{kj} \int_o^Z \kappa(L,z) u^m(z;r_j,t_k)dz \\ &= \sum_{jk} \phi^*_{jk} \overline{u^m}^Z(r_j,t_k) \\ &= \sum_{jk} \phi^*_{jk} \bar{u}^Z(r_j,t_k) + \sum_{jk} \phi^*_{jk} \bar{\varepsilon}^Z(r_j,t_k) \end{aligned} \qquad (5.4)$$

As the statistical properties of $\bar{u}^Z$ and $u^m$ may be somewhat different, $\phi_{jk} \neq \phi^*_{jk}$. However, as experience both from synoptic and mesoscale data indicate that there is no set of weight functions $\phi_{jk}$ that provides significantly better prediction than the next best, (Eidsvik 20,21), differences between $\phi_{jk}$ and $\phi^*_{jk}$ is not expected to have large effects on the prediction accuracy, at least not when discussing large scale effects. A useful approximation may therefore be

$$\hat{\bar{u}}^Z(r_o,t_o) \simeq \hat{\bar{u}}^Z(r_o,t_o) \qquad (5.5)$$

With the use of Equation (3.11), (5.4) and (5.5) we obtain

$$\hat{\bar{u}}^Z(r_o,t_o) \simeq \hat{\bar{u}}^L(r_o,t_o) + \widehat{u^{Tb}} \qquad (5.6)$$

$$\widehat{u^{Tb}} = \Sigma\phi^{*}_{jk}(u^{T} + \bar{\varepsilon}^{Z}) \tag{5.7}$$

The process output error is then

$$x(L) \approx \frac{\partial x}{\partial u}[\{\bar{u}^{L}(r_o,t_o) - \hat{u}^{L}(r_o,t_o)\} + \widehat{u^{Tb}}] \tag{5.8}$$

$$E\nu^2 = \sum_{j=1}^{M} (\sigma_{bj})^{-2} \sum_{p} \left(\frac{\partial x_j}{\partial u_p}\right)^2 [E\{\bar{u}^{L}_{p}(r_o,t_o) - \hat{u}^{L}_{p}(r_o,t_o)\}^2 + E\{\widehat{u^{Tb}}\}^2] \tag{5.9}$$

Under the hypotesis H5, the $u^{Tb}$ - term will be important only if $||r_i,t_k) - (r_p,t_q)|| = O(\Delta r,\Delta t)$ and/or a significant height - coherert measurement error. In the case of a small measurement error and $||r_i,t_j) - (r_p,t_g)|| >> O(\Delta r,\Delta t)$ the process output error becomes proportional to the prediction error of the normal atmospheric field.

$$x(L) \simeq \frac{\partial x}{\partial u}[u(r_o,t_o) - \hat{u}(r_o,t_o)] \tag{5.10}$$

At this stage it is possible to reexamine the neccessity of predicting the fields to be locally constant over intervals $(\Delta r,\Delta t)$. By varying $r_o$ and $t_o$ (and $\phi_{ij}$) in equation (5.2) a surface of the best predicted field would be obtained.

If this surface shows significant variation over distances of the order $(\Delta r,\Delta t)$, given by Equation (3.16), this prediction is still probably not much more accurate than the simple local prediction. It may even turn out that the benefit of "optimal" non-local predition method may be marginal (19, 20, 21).

## 6 REMARKS

Tradition in meteorology is to be interested and discuss extensively how to produce a predition of flow variables. However, it is the users benefit of small prediction error that enables the quality of the meteorological effort to be measured. With proper interpretation of variables. Figure 2.2 illustrates quite generally the meteorological problem of finding a near optimal prediction method for a particular process. We have given a broad outline of how this can be discussed for a particular class of processes.

Some convenient approximations and idealizations are suggested. They need not be valid for all combinations of system and weather parameters. It is sufficient that they are so in a representative subspace of system parameter phase space for most weather. If for instance linearized system equations are good approximations over large intervals of atmospheric variations in a representative subspace of system parameter space and not in the complementary subspace, we are satisfied to develop near optimal prediction method for the simplest case and also use these methods for the analytically difficult cases.

## 7 ACKNOWLEGEMENTS

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Noen systemers avhengighet av sine atmosfæriske omgivelser og varslingen av omgivelsene er drøftet.
Metoder for å oppnå tilstrekkelig nøyaktige varslingsmetoder for disse systemer er skissert.

TITLE On near optimal meteorological prediction methods for the prediction of trajectories through stochastic flow fields

ABSTRACT (max. 300 characters, 5-10 lines)

The dependence of some physical processes upon environmental flow and flow prediction is approached. Procedures for obtaining sufficiently accurate prediction methods for these processes are outlined.

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